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Non-viability of gravitational theory based on a quadratic lagrangian

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Abstract. It is shown that physically reasonable solutions of the field equations based on an R^2 lagrangian are possible. (R is the scalar curvature.) However, it is shown that experimental predictions of such a theory are at variance with observations. The most general quadratic lagrangian is also considered and it is shown that the R^2 term must dominate thus invalidating gravitational equations based on a general quadratic lagrangian.

1. Introduction

Field equations based on a lagrangian quadratic in the curvature tensor have had a long history in the theory of general relativity. Eddington (1923) mentioned the possibility of gravitational equations based on the quadratic invariants $R_{ij}R^{ij}$ and $R_{ijkl}R^{ijkl}$. Lanczos in a series of papers dating from 1932 (Lanczos 1932, 1938, 1949, 1957, 1962, 1963, 1966, 1967, 1969) has advocated quadratic lagrangians as a basis for a unified field theory of gravitation and electromagnetism. Also, quadratic terms appear as corrections to the Einstein lagrangian R when one takes into account the effects of vacuum polarization. This area has been fairly extensively studied (De Witt 1964, Michel 1973, Nariai 1971, Nariai and Tomita 1971, Pechlaner and Sexl 1966, Ruzmaikina and Ruzmaikin 1970), but in the present paper we shall not examine quadratic lagrangians in this context, nor from the point of view envisaged in Lanczos' work. We shall examine the viability of gravitational theory alone based on a lagrangian which is purely quadratic in the curvature tensor. Such a lagrangian does have some appeal: its similarity to the lagrangians of other field theories, for example, the electromagnetic or meson field, is obvious. In addition, the Schwarzschild solution is a solution of the resultant empty space equations. It might be argued, therefore, that the three crucial tests of general relativity do not eliminate equations based on a quadratic lagrangian from the class of possible gravitational equations. We shall see, however, that when one incorporates a stress–energy tensor into the theory then the observational consequences differ quite markedly from those of general relativity. This aspect of quadratic lagrangians has in fact been examined by Folomeshkin (1971). He concluded that in such a theory it was necessary for the trace of the energy–momentum tensor to be zero. He also concluded that if this were the case then the three crucial tests of relativity would be satisfied. It will be seen that Folomeshkin's first conclusion is not correct provided that one is prepared to admit (i) a cosmological constant and (ii) the constant coupling geometry to matter proportional to the cosmological constant. Even if we make these admissions then two of the three crucial tests of general relativity are not satisfied. This result finally eliminates quadratic lagrangian theories from the class of viable gravitational theories.

In this paper the signature of space–time is taken to be $(1, 1, 1, -1)$. This and other conventions are the same as that given in Synge (1966) with one exception: We use MKS units rather than Synge's geometrized units in which $G = C = 1$. The quantity κ is defined to be G/C^4.

2. The most general quadratic lagrangian

There are four algebraically independent invariants of the Riemann–Christoffel tensor, namely,

$$I_1 = R^2 \qquad I_2 = R_{ij}R^{ij} \qquad I_3 = R_{ijkl}R^{ijkl} \qquad I_4 = R_{ijkl}{}^*R^{ijkl},$$

${}^*R^{ijkl}$ being the left-handed dual of R^{ijkl}.

Only two of the above invariants are variationally independent, the following identities holding (Lanczos 1938):

$$\delta \int I_4\sqrt{-g}\, \mathrm{d}^4x = 0$$

and

$$\delta \int (I_3 - 4I_1 + I_2)\sqrt{-g}\, \mathrm{d}^4x = 0.$$

Accordingly we adopt as the most general quadratic lagrangian a linear combination of I_1 and I_2, ie,

$$L = \alpha R_{ij}R^{ij} + \beta R^2,$$

where α and β are constants. Matter is incorporated into the theory by a term M in the lagrangian so that the field equations are:

$$\delta \int (\alpha R_{ij}R^{ij} + \beta R^2 + \gamma M)\sqrt{-g}\, \mathrm{d}^4x = 0.$$

The stress energy tensor T_{ij} is the hamiltonian derivative of M with respect to the g^{ij}.

3. Equations of motion of test particles

Since the tensor T_{ij} is a hamiltonian derivative, it satisfies the conservation equations

$$T^j_{i;j} = 0.$$

For unstressed matter ($T_{ij} = \mu c^2 V_i V_j$) the above equations imply geodesic motion. For a test particle we assume that stress energy is negligible compared to the density and that test particles therefore follow geodesics of space–time as in other gravitational theories such as Einstein's theory or the Brans–Dicke theory.

4. Weak-field approximation and newtonian limit

In view of the extensive use of the weak-field approximation in the following sections we make a few preliminary statements concerning this approximation here.

If one takes the metric to be

$$g_{ij} = \eta_{ij} + h_{ij}$$

(η_{ij} = Minkowski tensor diag (1, 1, 1, −1)) and defines

$$\gamma_{ij} = h_{ij} - \tfrac{1}{2}\eta_{ij}\eta^{kl}h_{kl},$$

then the Einstein tensor is, to first order in γ_{ij}

$$G_{ij} = \tfrac{1}{2}(\Box^0\gamma_{ij} - \xi_{i,j} - \xi_{j,i} + \eta_{ij}\eta^{kl}\xi_{k,l}), \tag{1}$$

where

$$\xi_i = \eta^{kl}\gamma_{ik,l} \tag{2}$$

and $\Box^0$ is the d'Alembertian operator of flat space–time (cf Synge 1966 p 193). Usually one adopts the coordinate conditions $\xi_i = 0$ but this will not always be the case in this paper. As is well known, in the case of small curvature, $\frac{1}{2}h_{44}$ closely approximates a newtonian potential, asymptotically approaching $\kappa mc^2/r$ (M = gravitational mass in MKS units). In regions occupied by matter $\frac{1}{2}h_{44}$ satisfies an equation approximating

$$\nabla^2(\tfrac{1}{2}h_{44}) = -4\pi\kappa\rho c^2$$

where ρ is the density (cf Synge 1966 p 181). These considerations enable us to form constraints on the parameters appearing in our equations.

5. The lagrangian R^2

We consider first of all the lagrangian with an R^2 term only. The empty-space field equations are

$$R_{ij} - \tfrac{1}{4}g_{ij}R = \frac{1}{R}(g_{ij}\Box R - R_{;ij}),$$

or equivalently

$$G_{ij} = -\tfrac{1}{4}Rg_{ij} + \frac{1}{R}(g_{ij}\Box R - R_{;ij}). \tag{3}$$

Contracting these equations one finds that R satisfies the scalar wave equation, namely,

$$\Box R = 0. \tag{4}$$

In view of the simple equation satisfied by R it is more convenient to consider equations (3) as Einstein-type equations with source terms derived from the scalar field R satisfying (4).

Buchdahl (1962) rejected the R^2 equations because of the non-existence of asymptotically flat solutions. Briefly we can see the reason for Buchdahl's result. A static spherically symmetric weak-field approximation implies that $R = \alpha + \beta/r$ (α and β constants). The term $-\frac{1}{4}Rg_{ij}$ in the field equations then gives rise to terms in the metric proportional to r and r^2 so that space–time is not asymptotically flat. Nevertheless it is still possible to indicate the existence of physically reasonable solutions to these equations. This is important in this context since one of our main conclusions is that the general quadratic lagrangian is dominated by the R^2 term. We first of all note that solutions of Einstein's equations with cosmological constant, ie,

$$G_{ij} = \Lambda g_{ij}, \tag{5}$$

are solutions of (3) and (4), ie, the R^2 equations. It is well known that there exist solutions of (5), for instance the de Sitter space–time with line element

$$ds^2 = \frac{1}{1-\frac{1}{3}\Lambda r^2} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) - (1-\tfrac{1}{3}\Lambda r^2)\, dt^2,$$

which are not asymptotically flat. It is then natural to ask whether solutions of the R^2 equations can be found which are asymptotic to a cosmological space–time satisfying (5) the Einstein cosmological equations. Since, in the space–time described by such solutions, the background value of R would be -4Λ we normalize R by defining $\phi = -R/4\Lambda$ and the R^2 equations become

$$G_{ij} = \Lambda\phi g_{ij} - \frac{1}{\phi}\phi_{;ij} \tag{6}$$

and

$$\Box\phi = 0. \tag{7}$$

Here ϕ is of order unity. The term $-\frac{1}{4}Rg_{ij}$ which led to asymptotic non-flatness is now essentially a cosmological term.

A clearer insight into these equations is obtained if one considers them in the space conformally related by

$$g'_{ij} = |\phi| g_{ij}.$$

They are

$$G'_{ij} = \Lambda g'_{ij} - \tfrac{3}{2}(\psi_{,i}\psi_{,j} - \tfrac{1}{2}g'_{ij}g'^{kl}\psi_{,k}\psi_{,l})$$

$$\Box'\psi = 0,$$

where $\phi = \ln|\psi|$. These, of course, are Einstein's equations with cosmological constant coupled to a zero-mass scalar field. Accordingly they should yield physically reasonable solutions which are asymptotic to, say, a de-Sitter space–time, and in which ψ tends to zero far from matter. The metric of the original space–time is given by

$$g_{ij} = e^{-\psi} g'_{ij}$$

and so that metric too should be asymptotic to a de-Sitter metric. Even so, when one incorporates matter into these equations, the resultant experimental predictions of bending of light and precession of perihelion do not agree with observations, as the following analysis shows.

The field equations

$$\delta \int (R^2 + \mu M)\sqrt{-g}\, d^4x = 0$$

where M is the matter lagrangian, are:

$$G_{ij} = \Lambda\phi g_{ij} + \frac{1}{\phi}(g_{ij}\Box\phi - \phi_{;ij}) - \frac{\mu}{8\Lambda\phi} T_{ij},$$

ϕ as before is $-R/4\Lambda$ and from the contraction of these equations satisfies

$$\Box\phi = \frac{\mu}{24\Lambda}T.$$

We take the coupling coefficient $\mu = 64\pi\kappa\Lambda$ and this enables us to display the equations in a form where their differences from Einstein's equations are clear

$$G_{ij} = -\frac{8\pi\kappa}{\phi}T_{ij}+\Lambda\phi g_{ij}+\frac{1}{\phi}(g_{ij}\Box\phi-\phi_{;ij})$$

$$\Box\phi = \frac{8\pi\kappa}{3}T.$$

When the cosmological term $\Lambda\phi g_{ij}$ is neglected we find that we effectively have the Brans–Dicke equations for the case of $\omega = 0$ (ω is the Brans–Dicke coupling constant (Brans and Dicke 1961)). Substituting $\omega = 0$ into the Brans–Dicke results for the bending of light and the precession of perihelion gives the following results:

Deflection of light $= \frac{3}{4}\times$ Einstein value

Precession of perihelion $= \frac{2}{3}\times$ Einstein value.

These results are far removed from recent observations. Experiments on light bending give $1{\cdot}04^{+0{\cdot}15}_{-0{\cdot}10}\times$ Einstein value (Muhleman *et al* 1970) and precession of perihelion experiments give $1{\cdot}005\pm0{\cdot}007\times$ Einstein value (Shapiro *et al* 1972).

We can only conclude that the above values are too widely at variance with observations for the field equations based on $L = R^2$ and coupled to matter in the above way to merit any further attention.

It is implicit in the above that the scalar curvature be of cosmological order so that reasonable solutions of the R^2 equations may be obtained. This fact is also of importance in the following section.

6. $L = R_{ij}R^{ij}-\nu R^2$

We turn now to consider the most general quadratic lagrangian which we may take to be $R_{ij}R^{ij}-\nu R^2$. We shall show, by the use of the weak-field approximation, that in such a lagrangian the R^2 term dominates. The results of the previous section consequently preclude the possibility of gravitational theory based on a quadratic lagrangian.

The field equations

$$\delta\int(R_{ij}R^{ij}-\nu R^2-\mu M)\sqrt{-g}\,\mathrm{d}^4x = 0$$

are

$$\Box G_{ij}+(1-2\nu)\Box Rg_{ij}+(2\nu-1)R_{;ij}-2G^{ak}R_{aijk}$$
$$+(2\nu-1)RG_{ij}+\tfrac{1}{2}[G_{kl}G^{kl}+(\nu-1)R^2]g_{ij} = \mu T_{ij} \qquad (8)$$

and the contraction of these equations is $(2-6\nu)\Box R = \mu T$.

A naive weak-field approximation (with coordinate condition $\xi_i = 0$) yields

$$\tfrac{1}{2}\Box^{0^2}\gamma_{ij}+(1-2\nu)\Box^0 R\eta_{ij}+(2\nu-1)R_{,ij} = \mu T_{ij}.$$

Because of the square of the d'Alembertian operator appearing here, the newtonian limit, described previously, obviously cannot be regained. On reconsideration of the field equations (8), however, one realizes that if the term $(2\nu-1)R$ is large enough so that the term $(2\nu-1)RG_{ij}$ is not relegated to second order in the weak-field approximation then a second order term proportional to $\Box^0\gamma_{ij}$ survives. If we divide the field equations by $(2\nu-1)R$ then they are exhibited in a form which shows their close relationship to the R^2 equations

$$G_{ij} = \frac{\mu}{(2\nu-1)R}T_{ij} - \frac{1}{2}\frac{\nu-1}{2\nu-1}Rg_{ij} + \frac{1}{R}(\Box Rg_{ij} - R_{;ij})$$
$$-\frac{1}{(2\nu-1)R}(\Box G_{ij} - 2G^{ak}R_{aijk} + \tfrac{1}{2}G^{kl}G_{kl}g_{ij}).$$

Because of the term $-\frac{1}{2}[(\nu-1)/(2\nu-1)]Rg_{ij}$, R once again needs to be of cosmological order and we once again put $R = -4\Lambda\phi$. We take the coupling constant $\mu = 32(2\nu-1)\Lambda\pi\kappa$, and also make the substitution $k^2 = 4(2\nu-1)$. k is taken to be real, for otherwise solutions of the field equations can be obtained representing waves travelling faster than light. Because of the assumption that $(2\nu-1)R$ is 'large' then the parameter k^2 is 'large'. The field equations are then

$$G_{ij} = -\frac{8\pi\kappa}{\phi}T_{ij} + \frac{2(\nu-1)}{2\nu-1}\Lambda\phi g_{ij} + \frac{1}{\phi}(\Box\phi g_{ij} - \phi_{;ij}) - \frac{1}{k^2\phi}(\Box G_{ij} - 2G^{ak}R_{aijk} + \tfrac{1}{2}G^{kl}G_{kl}g_{ij}).$$

The R^2 equations are regained in the limit as ν (and hence k) tends to infinity.

7. Weak-field solution for a homogeneous sphere

With laboratory experiments in mind we determine the field of a homogeneous sphere. This enables us to determine a lower limit for k. We approximate ϕ by $1+\chi$ and neglect terms of order χ^2.

The weak field equations are

$$\Box^{0^2}\gamma_{ij} - \Box^0\xi_{i,j} - \Box^0\xi_{j,i} + \eta_{ij}\eta^{kl}\Box^0\xi_{k,l}$$
$$= k^2(\Box^0\gamma_{ij} - \xi_{i,j} - \xi_{j,i} + \eta_{ij}\eta^{kl}\xi_{k,l}) + k^2(\chi_{,ij} - \Box^0\chi\eta_{ij}) + 16\pi k_\kappa^2 T_{ij} \qquad (9)$$

with χ satisfying

$$\Box^0\chi = 8\pi\kappa\frac{2\nu-1}{6\nu-2}T, \qquad (10)$$

ξ_i as defined by equation (2) is $\eta^{kl}\gamma_{ik,l}$. Instead of the usual coordinate condition $\xi_i = 0$ we take

$$\Box^0\xi_i = k^2(\xi_i - \chi_{,i})$$

and the field equations become

$$\Box^{0^2}\gamma_{ij} = k^2\Box^0\gamma_{ij} + 16\pi k^2\left(T_{ij} - \frac{2\nu-1}{2(6\nu-2)}T\eta_{ij}\right), \qquad (11)$$

where we have made use of (9) the equation for χ.

For a homogeneous sphere with negligible stress energy $T_{ij} = \mu c^2 V_i V_j$ for $r \leqslant r_0$ where r_0 is the radius of the sphere and μ, the density, is constant.

We take for the line element

$$ds^2 = A(r)\,\delta_{\alpha\beta}\,dx^\alpha\,dx^\beta - B(r)(dx^4)^2$$

where A and B and their derivatives up to the third order are continuous. These conditions are contrary to the usual continuity conditions pertaining to general relativity but are appropriate for fourth-order equations.

The external solutions which are subsequently obtained for $h_{ij}\,(=g_{ij}-\eta_{ij})$ are

$$h_{\alpha\beta} = \frac{4\nu-1}{3\nu-1}\kappa mc^2\left(\frac{1}{r}-\frac{3}{k^3 r_0^3}(kr_0\cosh kr_0-\sinh kr_0)\frac{e^{-kr}}{r}\right)\delta_{\alpha\beta}$$

$$h_{44} = \frac{8\nu-3}{3\nu-1}\kappa mc^2\left(\frac{1}{r}-\frac{3}{k^3 r_0^3}(kr_0\cosh kr_0-\sinh kr_0)\frac{e^{-kr}}{r}\right).$$

We can see that, by absorbing the factor $\frac{1}{2}(8\nu-3)/(3\nu-1)$ into κ the above solution corresponds to the newtonian approximation, if k is 'large'. To determine how large k needs to be consider the gravitational 'force' $d(\frac{1}{2}h_{44})/dr$ which may be expressed as

$$\frac{8\nu-3}{2(3\nu-1)}\kappa mc^2\left[-\frac{1}{r^2}+3\left(\frac{(kr_0-1)+(kr_0+1)\,e^{-2kr_0}}{2k^3 r_0^3}\right)\left(\frac{1}{r^2}+\frac{k}{r}\right)e^{-k(r-r_0)}\right].$$

There are two significant components of this force which are manifest at a distance from the sphere, $(r-r_0)$, of order $1/k$. The first is a $1/r$ component of the force proportional to $e^{-k(r-r_0)}/r$ and the second is an additional $1/r^2$ term which at distances of order $1/k$ effectively renormalizes the gravitational constant. We conclude on the basis of this analysis that laboratory experiments for the determination of the gravitational constant constrain $1/k$ to be less than 1 cm. A typical value for the cosmological constant Λ is $10^{-56}\,m^{-2}$ (Adler *et al* 1965, p 367) and this, in view of the relationship between ν, k and Λ implies that ν is of the order of 10^{60}.

8. Static solution for an arbitrary source

The factor $\frac{1}{2}(2\nu-1)/(6\nu-2)$ appearing in (11), the final form of the weak-field equations, is effectively $\frac{1}{6}$, owing to the approximate value of ν obtained above. For static space–time with a perfect fluid source, equations (10) are

$$\nabla^4\gamma_{ij} = k^2\nabla^2\gamma_{ij}+16\pi k^2 S_{ij},$$

where

$$S_{ij} = (\mu c^2+p)V_i V_j+(\tfrac{1}{2}p+\tfrac{1}{6}\mu c^2)\eta_{ij}$$

and $V_i = \delta_{14}$ is the four-velocity of matter.

The Green function for the above set of equations is

$$\frac{1}{k^2R}-\frac{1}{k^2}\frac{e^{-kR}}{R}$$

where R is the distance from source point to field point so that the solution of the above equations is

$$\gamma_{ij} = -4\int_V S_{ij}(x_0^\alpha)\left(\frac{1}{R}-\frac{e^{-kR}}{R}\right) dv_0 .$$

The subscript 0 refers to the source point. This solution can be split into two parts

$$\gamma_{ij}^1 = -4\int \frac{S_{ij}(x_0^\alpha)}{R} dv_0$$

and

$$\gamma_{ij}^2 = 4\int S_{ij}(x_0^\alpha)\frac{e^{-kR}}{R} dv_0 .$$

γ_{ij}^1 is clearly the solution of these equations as k tends to infinity so that γ_{ij}^1 is the solution of the R^2 equations. By standard means we can show that

$$|\gamma_{ij}^2| \leqslant \frac{e^{-k(r-r_0)}}{r-r_0}\int |S_{ij}| dv_0 ,$$

where r_0 is the maximum value of the radial coordinate of the surface of the distribution of matter, the origin of the coordinate system being inside the source. With the value of k determined previously, γ_{ij}^2 will be insignificant for values of r greater than one centimetre. Thus the weak-field solutions of the general quadratic lagrangian equations exhibit the same characteristics as the equations of the more specialized R^2 theory when the distance from the source is much greater than a centimetre. We surmise that the same will be true for exact solutions of the quadratic lagrangian equations. We may add weight to this supposition by noting that the extra terms resulting from the more general quadratic lagrangian vanish exponentially. A post-newtonian approximation would still involve these exponentially vanishing terms. For the purposes of solar system tests of the theory the effects of the more general equations and that of the R^2 theory are indistinguishable.

9. Conclusions

There are strong indications that the field equations based on the lagrangian R^2 rather than the standard R possess physically reasonable solutions. However, the introduction of a matter term into the lagrangian gives rise to theoretical predictions which are at variance with observation. This conclusion is not altered if one employs the most general quadratic lagrangian, namely $R_{ij}R^{ij}-\nu R^2$, since the resulting solutions of the equations for the latter theory differ insignificantly from those of the R^2 theory. These results eliminate gravitational theories based on quadratic lagrangians from the realm of viable gravitational theories and point strongly towards the uniqueness of the Einstein equations.

References

Adler R, Bazin M and Schiffer M 1965 *Introduction to General Relativity* (New York: McGraw-Hill)
Brans C and Dicke R H 1961 *Phys. Rev.* **124** 925
Buchdahl H A 1962 *Nuovo Cim.* **123** 141
De Witt B S 1964 *Relativity Groups and Topology* ed C De Witt and B De Witt (New York: Gordon and Breach)
Eddington A S 1923 *The Mathematical Theory of Relativity* (Cambridge: Cambridge University Press)
Folomeshkin V N 1971 *Commun. math. Phys.* **22** 115
Lanczos C 1932 *Phys. Rev.* **39** 716
—— 1938 *Ann. Math.* **39** 842
—— 1949 *Rev. mod. Phys.* **21** 497
—— 1957 *Rev. mod. Phys.* **29** 337
—— 1962 *Rev. mod. Phys.* **34** 379
—— 1963 *J. math. Phys.* **4** 316
—— 1966 *J. math. Phys.* **7** 951
—— 1967 *J. math. Phys.* **8** 829
—— 1969 *J. math. Phys.* **10** 1059
Michel F C 1973 *Ann. Phys., NY* **76** 281
Muhleman D O *et al* 1970 *Phys. Rev. Lett* **24** 1377
Nariai H 1971 *Prog. theor. Phys.* **46** 433
Nariai H and Tomita K 1971 *Prog. theor. Phys.* **46** 776
Pechlaner E and Sexl R 1966 *Commun. math. Phys.* **2** 165
Ruzmaikina T V and Ruzmaikin A A 1970 *Sov. Phys.–JETP* **30** 372
Shapiro I I *et al* 1972 *Phys. Rev. Lett.* **28** 1594
Synge J L 1966 *Relativity: the General Theory* (Amsterdam: North Holland)